88.70 Some Impossible Constructions in Elementary Geometry

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$$DC \times DB = DF \times DI_3 \Rightarrow s(s-a) = r_2 r_3 = \frac{\Delta}{(s-b)} \cdot \frac{\Delta}{(s-c)}$$

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

Appendix

$$(r_2 r_3) \times (r r_1) = s(s-a) \times (s-b)(s-c) \Rightarrow \Delta = \sqrt{r r_1 r_2 r_3}.$$

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88.70 Some impossible constructions in elementary geometry

Undergraduate abstract algebra textbooks usually include a discussion of the three classical constructions: trisecting an angle, doubling the cube, and squaring the circle, by straight-edge and compass alone (see, e.g., [1, 2, 3, 4]). Indeed, such a discussion demonstrates how the purely algebraic concept of extension fields can be used to settle in the negative three geometric problems that remained open for over 2000 years. In this note we show, in a way accessible to undergraduate mathematics students, that the following construction problem (*) is likewise impossible:

> Given a non-circle conic and any point P in its plane, to construct by straight-edge and compass alone the line PM where M is the point on the conic whose distance from P is minimum.

Of course, if the conic were a circle, the construction of PM would be straightforward, and so the problem illustrates an elementary aspect of circles that is not generally shared by ellipses. (Another aspect that general ellipses do not share with circles is the computation of the perimeter: while this is easy for circles, the case of ellipses is an entirely different story, and, as is well known, involves such advanced concepts as elliptic integrals.) It is worth mentioning perhaps that the impossibility of our minimisation problem (*) follows, in the case of the ellipse or the hyperbola, from the impossibility of the first classical problem (angle trisection), while the case of the parabola follows from the second classical problem (cube duplication). It may also be of pedagogical interest to note that the arguments we use are very elementary, and bring together basic notions from algebra, calculus, geometry and trigonometry. For some similar straight-edge and compass constructions, we refer to [5] and [6] on Al-Hazen's Problem, and to [7] on the construction of minimal chords in a parabola.

A real number is *constructible* if it can be obtained from rational numbers by means of a finite number of operations, each of which is one of the operations $+$, $-$, $\times$, $\div$ of elementary arithmetic, or extraction of a square root. For details the reader should consult any of the first four references below. In particular, we will make use of the following definitions and propositions (see [3, pp. 544-546]):

1. The set of constructible numbers is a subfield of $\mathbb{R}$.
2. A point (a, b) is constructible (or located) in the Euclidean plane if a and b are constructible.
3. A line segment is constructible in the Euclidean plane if its endpoints are constructible.
4. An angle θ is defined to be constructible if $\cos\theta$ (or, equivalently, $\sin\theta$) is constructible.

   (Note that the angle whose radian measure is π is constructible although the number π itself is not, since circles cannot be squared.)
5. The angle $\pi/9$ is not constructible (i.e. the angle $\pi/3$ cannot be trisected).
6. $\sqrt[3]{2}$ is not constructible (i.e. cubes cannot be doubled).

Let us first show that there exists an ellipse with constructible major and minor axes and a constructible point in the plane of the ellipse for which problem (*) is impossible. Suppose that we set up an xy-coordinate system so that the ellipse is centered at the origin, its major axis is horizontal and has length $2a$, its minor axis is vertical and has length $2b$, the given point P with coordinates (r, s) is in the first quadrant (so the closest point on the ellipse to P is in the first quadrant), and a, b, r, s are constructible real numbers. The ellipse has equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and, if M is any point (x, y) on the ellipse, and $f(x)$ is the square of the distance PM then

$$f(x) = (x - r)^2 + (y - s)^2 = (x - r)^2 + \left(\frac{b\sqrt{a^2 - x^2}}{a} - s\right)^2,$$

$(0 < x < a)$. We take the positive root, since M is in the upper half plane. For the minimum of f to occur at x, we must have $f'(x) = 0$. i.e.

$$\frac{r}{x} - \left(1 - \frac{b^2}{a^2}\right) = \frac{bs}{a\sqrt{a^2 - x^2}}.$$

It follows that $0 < x < a$. Put $x = a\cos\theta$, $0 < \theta < \pi/2$, to get

$$\left(a^2 - b^2\right)\sin\theta\cos\theta = ar\sin\theta - bs\cos\theta. \qquad (1)$$

(Note that when $a = b$, i.e. in case of a circle, $\tan\theta = s/r$, and the required construction of the line PM is straightforward.)

If the line PM could be constructed by straight-edge and compass, then x, and therefore θ, would be constructible. Suppose we can choose a, b, r, s so that

$$\cos\alpha = \frac{2ar}{a^2 - b^2} \text{ and } \sin\alpha = \frac{2bs}{a^2 - b^2}$$

for some α in $(0, \pi/2)$, then (1) yields $\sin 2\theta = \sin(\theta - \alpha)$, and so, either $2\theta = \theta - \alpha + 2k\pi$ or $2\theta = -(\theta - \alpha) + (2k + 1)\pi$, for some integer k. Since both θ and α are in the first quadrant, we obtain $2\theta = \pi - (\theta - \alpha)$, i.e.

$\theta = \frac{\pi}{3} + \frac{\alpha}{3}$. Thus $\alpha / 3$ would be constructible. It is now easy to check that if $a = 3$, $b = 1$, $r = 2/3$, $s = 2\sqrt{3}$, then $\cos\alpha = \frac{1}{2}$ and $\sin\alpha = \frac{\sqrt{3}}{2}$, i.e. $\alpha = \pi / 3$. But we know that the angle $\pi / 3$ cannot be trisected, contradicting that $\alpha / 3$ is constructible. This means that it is impossible to locate (in the sense explained in [1] and [3]) the point on the ellipse $\frac{x^2}{9} + y^2 = 1$ nearest to the point $P(2/3,\ 2\sqrt{3})$, or, equivalently, the normal lines from the point $(2/3,\ 2\sqrt{3})$ to the ellipse $\frac{x^2}{9} + y^2 = 1$ are not constructible.

Starting with the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ of a hyperbola where a and b are constructible real numbers, an argument similar to the one above (using $x = a \sec\theta$) shows that it is impossible to locate the point on the hyperbola $x^2 - y^2 = 1$ nearest to the point $(8,\ 2\sqrt{3})$.

We next show that there exists a parabola with a horizontal constructible directrix and constructible focus for which problem (*) is impossible. As in the case of the ellipse, suppose that we set up an xy-coordinate system so that the vertex of the parabola is at the origin, its directrix has equation $y = -p$, and its focus is at $(0, p)$, where p is a positive constructible number. The parabola then has equation $4py = x^2$, and for any points $P(r, s)$ in the plane and $M(x, y)$ on the parabola, the square of the distance PM is $g(x) = (x-r)^2 + (y-s)^2 = (x-r)^2 + \left(\frac{x^2}{4p} - s\right)^2$. If the minimum of g occurs at x then $g'(x) = 0$, i.e. $x^3 + 4p(2p - s)x - 8p^2r = 0$. The choice $p = \frac{1}{2}$, $r = s = 1$, leads to the equation $x^3 - 2 = 0$, which, as we know, does not have a constructible root. This means that it is impossible to locate the point on the parabola $2y = x^2$ nearest to the point $(1,\ 1)$ by straightedge and compass alone, or, equivalently, the normal line from the point $(1,\ 1)$ to the parabola $2y = x^2$ is not constructible.

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88.71 Aubel's theorem revisited

Let P, Q, R, S be the centres of the outward facing squares attached to sides AB, BC, CD, DA of a general quadrilateral $ABCD$, respectively.

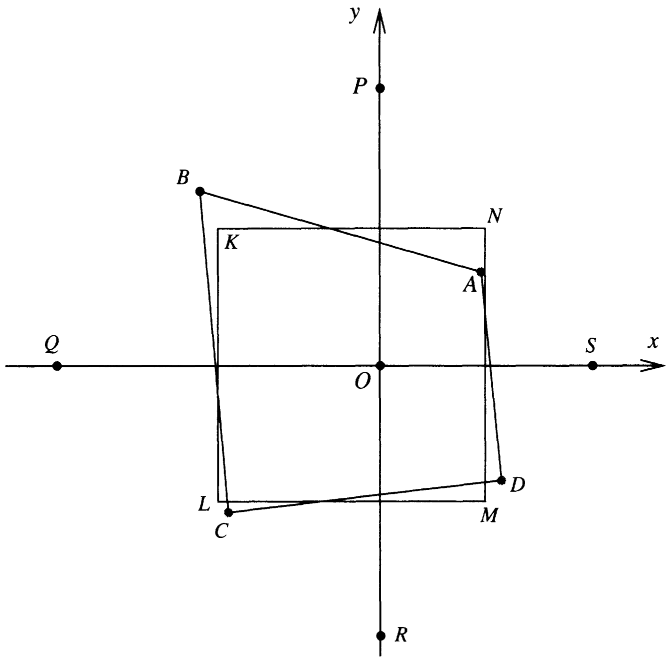

FIGURE 1

The theorem states that PR is perpendicular and equal to QS (see [1]).

So, how large is PR relative to $ABCD$? To answer this question we first attach the points P, Q, R, S to the coordinate axes, taking $P = P(0, p)$, $Q = Q(-q, 0)$, $R = R(0, -r)$ and $S = S(s, 0)$. We also take $A = A(x_a, y_a)$, $B = B(x_b, y_b)$, $C = C(x_c, y_c)$, $D = D(x_d, y_d)$.

Now to find the equations relating the coordinates of the points A, B, P, say, we may suppose without loss of generality that A and B lie in the first quadrant as shown in Figure 2 below. When this is not the case, it is easy to check that the equations which follow are still correct.